Solution to Math4230 Tutorial 6

1. Let C be a nonempty closed convex subset of $\mathbb{R}^{n+1}$ that contains no vertical lines. Show that C is equal to the intersection of the closed halfspaces that contain it and correspond to nonvertical hyperplanes.

   Solution

   C is contained in the intersection of the closed halfspaces that contain C and correspond to nonvertical hyperplanes, so we focus on proving the reverse inclusion. Let $x \notin C$. By assumption C does not contain any vertical lines, we can apply the nonvertical hyperplane theorem and we see that there exists a closed halspace that correspond to a nonvertical hyperplane, containing C but not containing x. Hence if $x \notin C$, then x cannot belong to the intersection of the closed halfspaces containing C and corresponding to nonvertical hyperplanes, proving that C contains that intersection.

2. Find out the following conjugate function of f
   (a) $f(x) = -\log x$
   (b) $f(x) = \frac{1}{2}x^T Qx$ with $Q \in \mathbb{R}^{n\times n}$ is symmetry positive define matrix and $x \in \mathbb{R}^n$

   Solution

   Hint: (a) when $y \geq 0$, the function $xy + \log x$ is unbounded increasing function. For the other case, just take the derivative of the function and compute the critical point.

   (a) $f^*(y) = \begin{cases} -1 - \log(-y) & \text{if } y < 0 \\ \infty & \text{otherwise} \end{cases}$

   (b) $f^*(y) = \frac{1}{2}y^T Q^{-1}y$

3. Given the conjugate function of g is g^*, compute the conjugate function of $f_1(x) = g(x) + a^T x + b$ and $f_2(x) = g(x - b)$.

   Solution

   (a) $f_1^*(y) = g^*(y - a) - b$;
   (b) $f_2^*(y) = b^T y + g^*(y)$.